Application of Bessel functions: Transient heat conduction in a thin wall pipe

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Introduction

Bessel functions routinely occur in the solution of PDEs defined on a domain described by cylindrical coordinates. In this tutorial we show how Mathematica can be used to help solve problems involving Bessel functions.

Problem Statement

A very long hollow cylinder of inner radius a and outer radius b is made of conducting material of diffusivity $\kappa$. If the inner and outer surfaces are kept at temperature zero while the initial temperature is given by the function $F(r)$, where $r$ is the distance from the axis, find the temperature at any point at any later time $t$. The geometry of the problem is illustrated in the following figure.

Solution

Since the geometry shows no $z$ or $\phi$ dependence the problem we must solve for the temperature distribution is

$$\frac{\partial u}{\partial t} = k \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right), \quad 0 < r < 1$$

IC: $u(r, 0) = F(r)$
BC1: $u(a, t) = 0$
BC2: $u(b, t) = 0$

Following the separation of variables procedure shown in the previous problem, the general solution to the eigenvalue problem

$$\frac{d^2 \phi}{dr^2} + \frac{1}{r} \frac{d\phi}{dr} + \lambda^2 \phi = 0, \quad \phi(a) = 0, \phi(b) = 0$$
is
\[ \phi (r) = c_1 J_0 (\lambda r) + c_2 Y_0 (\lambda r) \]

From \( \phi (a) = 0, \phi (b) = 0 \) we find
\[ c_1 J_0 (\lambda a) + c_2 Y_0 (\lambda a) = 0, \quad c_1 J_0 (\lambda b) + c_2 Y_0 (\lambda b) = 0 \quad (1) \]

This represents two homogeneous equations for the constants \( c_1 \) and \( c_2 \). We can write this system of equations as
\[
\begin{pmatrix}
J_0 (\lambda a) & Y_0 (\lambda b) \\
J_0 (\lambda b) & Y_0 (\lambda b)
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix} = 0
\]

For a non trivial solution to exists, the determinant
\[ \det \begin{pmatrix}
J_0 (\lambda a) & Y_0 (\lambda b) \\
J_0 (\lambda b) & Y_0 (\lambda b)
\end{pmatrix} = 0 \]

This gives
\[ Y_0 (\lambda a) J_0 (\lambda b) - J_0 (\lambda a) Y_0 (\lambda b) = 0 \]

which is the equation for determining \( \lambda \). This equation has infinitely many roots \( \lambda_1, \lambda_2, \lambda_3, \ldots \)

From the first equation in (1) we can solve for \( c_2 \) in terms of \( c_1 \)
\[ c_2 = -\frac{c_1 J_0 (\lambda a)}{Y_0 (\lambda a)} \]

Thus our eigenfunction can be represented by
\[ \phi_m (r) = Y_0 (\lambda_m a) J_0 (\lambda_m r) - J_0 (\lambda_m a) Y_0 (\lambda_m r) \]

The general solution can then be written as
\[ u (r, t) = \sum_{m=1}^{\infty} A_m e^{-k \lambda_m^2 t} \phi_m (r) \]

with
\[ \phi_m (r) = Y_0 (\lambda_m a) J_0 (\lambda_m r) - J_0 (\lambda_m a) Y_0 (\lambda_m r) \]

Note that since our eigenvalue problem is a Sturm Liouville system, we are assured that the eigensolutions \( \phi_m \) are orthogonal to \( \phi_n \) with respect to the weight \( \sigma (r) = r \) i.e.,
\[ \int_a^b \phi_m (r) \phi_n (r) r \, dr = 0 \quad \text{if } m \neq n \]

Applying the orthogonality condition to the IC condition relationship
\[ u (r, 0) = F (r) = \sum_{m=1}^{\infty} A_m \phi_m (r) \]

gives
\[ A_m = \frac{\int_a^b r F (r) \phi_m (r) \, dr}{\int_a^b r \phi_m^2 (r) \, dr} \]

Let us try to see what the eigenvalues look like for this problem. First consider the Bessel function of the Second Kind of order zero, viz., \( Y_0 (r) \). Mathematica has a built-in function for \( Y_0 \) called \textbf{BesselY[0,r]}.

Here is a plot of it
Next let us take \( a = 1, \quad b = 2, \) and then the equation for the eigenvalues is given
\[
f (\lambda) \equiv Y_0 (\lambda) J_0 (2 \lambda) - J_0 (\lambda) Y_0 (2 \lambda) = 0
\]

Here is a plot of this function
\[
eqn = BesselY[0, \lambda] BesselJ[0, 2 \lambda] - BesselJ[0, \lambda] BesselY[0, 2 \lambda];
Plot[eqn, \{\lambda, 0, 40\}, Frame \rightarrow True, FrameLabel \rightarrow \{"r", "f(\lambda)"\}]
\]

Thus the first zero is near 3, the next near 6, the next near 9 etc. We can use \texttt{FindRoot} to determine the roots. Here is the first root \( \lambda_1 \)
\[
\text{FindRoot}[eqn == 0, \{\lambda, 3\}]
\]
\[
\{\lambda \rightarrow 3.12303\}
\]

The following program will calculate as many roots as desired. Please consult the ECH198 notebooks for more details.
Clear[root, guess, rootlist];
guess = 3; rootlist = {}; n = 20;
Do[{root = FindRoot[Evaluate[eqn == 0], {λ, guess}];
AppendTo[rootlist, root[[1, 2]]];
guess = 3 + root[[1, 2]], {i, 1, n}];
rootlist


We can also use Mathematica to test whether the eigenfunctions we found are orthogonal over the interval $r = 1$ to $r = 2$. Let us take $\phi_1$ and $\phi_3$ for our test case. Thus we must show that

$$
\int_1^2 \phi_1 (r) \phi_3 (r) r \, dr = 0
$$

with

$$
\phi_m (r) = Y_0 (\lambda_m a) J_0 (\lambda_m r) - J_0 (\lambda_m a) Y_0 (\lambda_m r)
$$

Here is the Mathematica code that carries out the test. First we define a function $\lambda[m]$ which extracts the eigenvalues from the list rootlist calculated previously.

$$
\lambda[m_] := rootlist[[m]]
$$

Thus, suppose we want the second eigenvalue

$$
\lambda[2] = 6.27344
$$

Next we define a function $\phi[m, \lambda, r]$ which computes the m-th eigenfunction

$$
\phi[m_, \lambda_, r_] := BesselY[0, \lambda[m]] BesselJ[0, \lambda[m] r] - BesselJ[0, \lambda[m]] BesselY[0, \lambda[m] r]
$$

Here is what $\phi_1$ looks like:

$$
\phi[1, \lambda, r] = 0.33499 BesselJ[0, 3.12303 r] + 0.298891 BesselY[0, 3.12303 r]
$$

Now suppose we are given $f (r)$, then the coefficients $A_m$ are determined by

$$
A_m = \int_a^b r f (r) \phi_m (r) \, dr / \int_a^b r \phi_m^2 (r) \, dr
$$

The following Mathematica program calculates the coefficients $A_m$

Clear[A]
f[r_] := -r (1 - r) (2 - r)
A[m_] := A[m] = NIntegrate[f[r] r, {r, 1, 2}] / NIntegrate[f[r] r, {r, 1, 2}]

The following evaluation of the coefficients takes awhile,

Table[A[m], {m, 1, 10}]

{-2.35751, 0.880954, -0.308691, 0.21773, -0.112521, 0.0965507, -0.0576058, 0.0542662, -0.0348972, 0.0347174}

The general solution to the transient problem is given by
We define the following function that determines the partial sum (m terms)

\[ u[r, t] = \sum_{m=1}^{\infty} A_m e^{-\lambda n^2 t} \phi_m(r) \]

Here is a plot of the solution at \( t = 0 \), with 10 terms

\[
\text{Plot}[\text{Evaluate}[u[r, 0, 10]], \{r, 1, 2\}, \text{Frame} \to \text{True}, \text{FrameLabel} \to \{"r", "u(r,0,10)"\}]
\]

The transient solution can be displayed with Plot3D

\[
\text{Plot3D}[\text{Evaluate}[u[r, t, 10]], \{t, 0, .3\}, \{r, 1, 2\}, \text{PlotPoints} \to 20, \text{PlotRange} \to \{0, .4\}]
\]

Finally, we use the function \texttt{NIntegrate} to test the validity of the orthogonality relationship

\[
\text{Quiet}[\text{NIntegrate}[\phi[1, \lambda, r] \phi[3, \lambda, r] r, \{r, 1, 2\}]] = -3.33392 \times 10^{-18}
\]
Thus as expected the eigenfunctions are orthogonal, to the accuracy of the numerical integration.